Proposition 5.7. *Consider a motion $\boldsymbol{\phi}(\mathbf{x}, t)$ with corresponding spatial velocity field $\mathbf{v}(\mathbf{x}, t)$. Let the spatial vector field $\mathbf{h}(\mathbf{x}, t)$ satisfy*

$$\frac{D}{Dt}\mathbf{h}(\mathbf{x}, t) = \Gamma(\mathbf{x}, t)\mathbf{h}(\mathbf{x}, t), \quad \mathbf{x} \in \Omega_t, \;\; \mathbf{h}(\mathbf{X}, 0) = \mathbf{h_0}(\mathbf{X}) \quad \mathbf{X} \in \Omega \, (= \Omega_0), \tag{5.6}$$

where $\Gamma = (\Gamma_{ij}) = \left(\frac{\partial v_i}{\partial x_j}(\mathbf{x}, t)\right)$ *is the velocity gradient tensor. Then*

$$\mathbf{h}(\mathbf{x}, t)|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{X},t)} = D\boldsymbol{\phi}(\mathbf{X}, t)\mathbf{h}_0(\mathbf{X}) = \mathbf{F}(\mathbf{X}, t)\mathbf{h}_0(\mathbf{X}), \tag{5.7}$$

where $F(\mathbf{X}, t) = D\boldsymbol{\phi}(\mathbf{X}, t) = \nabla_{\mathbf{X}}\boldsymbol{\phi}(\mathbf{X}, t)$ *is the deformation gradient.*

Proof. We first show that $\bar{\mathbf{h}}(\mathbf{x}, t)$ defined by

$$\bar{\mathbf{h}}(\mathbf{x}, t)\big|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{X},t)} = F(\mathbf{X}, t)\mathbf{h_0}(\mathbf{X})$$

satisfies the given equation:

$$\left(\frac{D}{Dt}\bar{\mathbf{h}}(\mathbf{x}, t)\right)\bigg|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{X},t)} = \frac{\partial}{\partial t}F(\mathbf{X}, t)\mathbf{h_0}(\mathbf{X}) = \Gamma(\mathbf{x}, t)|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{X},t)}\, F(\mathbf{X}, t)\mathbf{h_0}(\mathbf{X}) = \Gamma(\mathbf{x}, t)\bar{\mathbf{h}}(\mathbf{x}, t)\big|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{X},t)}.$$

To see that this is the unique solution, notice that if $\hat{\mathbf{h}}(\mathbf{x}, t)$ is another solution then $\hat{\mathbf{h}}(\boldsymbol{\phi}(\mathbf{X}, t), t)$ satisfies the same first order system of ordinary differential equations in t (viewing $\mathbf{X}$ as a parameter) and the same initial conditions. Since solutions to such an initial value problem are unique, the result follows. □

Corollary 5.8. *(Cauchy's Theorem on transport of vorticity) Suppose further that ~~that~~ the velocity field $\mathbf{v}(\mathbf{x}, t)$ satisfies the incompressible Euler Equations, then the vorticity satisfies*

$$\boldsymbol{\omega}(\mathbf{x}, t)|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{X},t)} = \nabla_{\mathbf{X}}\boldsymbol{\phi}(\mathbf{X}, t)\boldsymbol{\omega}_0(\mathbf{X}) = F(\mathbf{X}, t)\boldsymbol{\omega}_0(\mathbf{X}), \quad F(\mathbf{X}, t) = D\boldsymbol{\phi}(\mathbf{X}, t). \tag{5.8}$$

Proof. This follows from Sheet 4 Q3 since the vorticity satisfies

$$\frac{D}{Dt}\boldsymbol{\omega}(\mathbf{x}, t) = (\boldsymbol{\omega}.\nabla)\mathbf{v}(\mathbf{x}, t) = \Gamma(\mathbf{x}, t)\boldsymbol{\omega}(\mathbf{x}, t).$$

□

Theorem 5.9. *(Helmholtz result on transport of vortex lines.) For solutions of the incompressible Euler-Equations, vortex lines are transported by the flow.*

Proof.

Let $\boldsymbol{\phi} : \Omega \times [0, T] \to \mathbb{R}^3$ be a motion. Let $\mathbf{y}(s)$, $\mathbf{y} : [a, b] \to \Omega$ be a vortex line in the fluid at time $t = 0$. Then, by time t, this curve has deformed to become the curve $\bar{\mathbf{y}}(s) = \boldsymbol{\phi}(\mathbf{y}(s), t)$, $s \in [a, b]$. Since

$$\begin{aligned} \frac{d\bar{\mathbf{y}}(s)}{ds} &= D\boldsymbol{\phi}(\mathbf{X}, t)|_{\mathbf{X}=\mathbf{y}(s)} \frac{d\mathbf{y}(s)}{ds} \\ &= D\boldsymbol{\phi}(\mathbf{y}(s), t)\boldsymbol{\omega}_0(\mathbf{y}(s)) \\ &= \boldsymbol{\omega}(\boldsymbol{\phi}(\mathbf{y}(s), t), t), \quad \text{(using Corollary 5.8)}, \\ &= \boldsymbol{\omega}(\bar{\mathbf{y}}(s), t) \end{aligned}$$

it follows that $\bar{\mathbf{y}}(s)$ is also a vortex line.

5.4 Steady irrotational flow in 2D, Complex Velocity

In this section we write $(x_1, x_2) = (x, y) \in \mathbb{R}^2$ and consider planar velocity fields of the form

$$\mathbf{v}(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \\ 0 \end{pmatrix}, \tag{5.9}$$

where u, v are sufficiently smooth functions.

Given a 2D velocity field, we form the complex velocity function

$$W(z) = u(x,y) - iv(x,y) \text{ for } z = x + iy \in \mathbb{C}. \tag{5.10}$$

Now consider the complex contour integral around the closed simple contour C parametrised by $s \in [a, b]$.

$$\int_C W(z)\, dz = \int_a^b (u - iv)\left(\frac{dx}{ds} + i\frac{dy}{ds}\right) ds = \int_a^b u\frac{dx}{ds} + v\frac{dy}{ds}\, ds + i\int_a^b u\frac{dy}{ds} - v\frac{dx}{ds}\, ds, \tag{5.11}$$

hence the real part of the integral is the circulation around the contour C and the imaginary part is related to the flow perpendicular to C. (Note that $\begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix}$ is normal to the contour C.

If we suppose that the flow is incompressible, then

$$0 = \nabla.\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial}{\partial y}(-v) \tag{5.12}$$

and if the flow is irrotational, then

$$\mathbf{0} = \nabla \times \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix} \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial}{\partial x}(-v) \tag{5.13}$$

and so the real and imaginary parts of the complex velocity satisfy the Cauchy-Riemann equations of complex analysis. Hence, $W(z)$ is a complex analytic function on its domain of definition.

Remark 5.10 (Velocity potential)**.** *If the flow is irrotational, then* $\nabla \times \mathbf{v} = 0$ *and so*

$$\mathbf{v}(x,y) = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ 0 \end{pmatrix}, \tag{5.14}$$

where $\phi(x,y)$ *is the velocity potential.*

If we suppose further that the flow is incompressible, then

$$0 = \nabla.\mathbf{v} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \Delta\phi, \tag{5.15}$$

and so ϕ *is harmonic.*

Remark 5.11 (Stream function). *Notice that if we do not assume that the flow is irrotational, then by a result from vector calculus, the incompressibility condition $\nabla . \mathbf{v} = 0$ implies the existence of a scalar function ψ called the stream function such that*

$$\nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \psi_y \\ -\psi_x \\ 0 \end{pmatrix}$$

and hence

$$\nabla \times \mathbf{v} = \nabla \times \begin{pmatrix} \psi_y \\ -\psi_x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\Delta\psi \end{pmatrix}$$

is the corresponding vorticity. In particular, if the flow is also irrotational, then ψ is harmonic.

Consider flow in a 2D domain external to D (corresponding to a body immersed in the flow). Consider a 2D velocity field and form the complex function If we paramatrise the boundary of D as $(x(s), y(s))$ using arc length $s \in [a, b]$ as the parameter, then the vector field

$$\begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix} \tag{5.16}$$

is the unit tangent vector to ∂D and hence

$$\begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix} \tag{5.17}$$

is the outward pointing normal unit vector to D on ∂D.

On the boundary of the body D there is no normal component of velocity and so

$$0 = \mathbf{v}.\mathbf{n} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} . \begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix} . \tag{5.18}$$

Now consider the complex contour integral

$$\int_C W(z)\, dz = \int_{\partial D} W(z)\, dz = \int_a^b (u - iv) \left(\frac{dx}{ds} + i\frac{dy}{ds} \right) ds \tag{5.19}$$

$$= \int_a^b u\frac{dx}{ds} + v\frac{dy}{ds}\, ds + i\int_a^b u\frac{dy}{ds} - v\frac{dx}{ds}\, ds = \text{Circulation around } C.$$